論　文

Population Dynamics, Longer Life Expectancy, and Child-Rearing Policies in an R&D-based Growth Model with Overlapping Generations*

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Abstract

This study constructs an overlapping generations model with R&D activities and endogenous fertility. We demonstrate that there is a trap region where the economy cannot achieve sustainable growth. Moreover, this study shows that an increase in life expectancy expands the region where R&D is conducted, but shrinks the region where population increases. We consider the effects of child-rearing policies to expand the region where population increases. We show that these policies have the opposite effect of increasing life expectancy if the tax burden effect is small. However, a strengthening of child-rearing policies can reduce both regions where R&D is conducted and where population increases if the tax burden effect is large.

JEL Codes：J13, O30, O41
Keywords：R&D, Fertility, Increasing life expectancy, Child-rearing policies

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R&D 活動を考慮した世代重複モデルにおける、人口増加、平均余命の上昇及び育児支援策の影響

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〈要 旨〉

本稿は R&D 活動と家計の出生選択を考慮した世代重複モデルを構築し、R&D による技術進歩と出生率の関係について分析を行った。主要な結果は以下の通りである。持続的な経済成長が不可能な領域が存在する。平均余命の上昇は R&D の行われる領域を上昇させる一方で、人口が増加する領域を縮小させる。人口が増加する領域を上昇させるために、本稿では育児支援策を拡充させたときの効果を分析し、以下の結果を得た。家計への税負担が小さいときは平均余命が上昇したときと異なり、R&D の行われる領域を縮小させて、人口が増加する領域を拡大させる。しかしながら、税負担が大きいときは R&D の行われる領域と人口が増加する領域がともに縮小してしまう。

JEL 分類コード：J13, O30, O41
キーワード：R&D、人口増加、平均余命の上昇、育児支援政策

1. Introduction

The Japanese economy is in a low-growth trap like many other developed countries. This is attributed to slowdown of technical progress. If firms do not adopt new technologies, do not develop new products, or do not use new production methods, technological advances decelerate. For example, Fukao (2012) indicates that low investment in information and communication technology is the main cause of the slowdown. What are the reasons for this? One answer is that the working population size in Japan has stagnated because of low fertility rates since the late-1970s. As the literature on endogenous growth shows, population size is one of the important determinants of technical progress. See, for example, Romer (1990), Grossman and Helpman (1991), and Jones (1995).

What induces parents to have fewer children? A high child-rearing cost is a possible reason. When couples have more children, they must incur the cost of rearing children by giving up some consumption. Therefore, if the income level is relatively low, a couple decreases the number of children. In addition, we must consider other factors that affect the fertility rate and the working population size. Improvements in medical and sanitary control standards increase life expectancy. This has resulted in population aging in almost all developed countries. In particular, in Japan, there has been a rapid increase in the ratio of older people to the working-age compared with other countries. With people having to save more to prepare for their old age, they decrease the number of children.

To examine the relation between fertility choice and technical progress, this study constructs an overlapping generations model with R&D activities and endogenous fertility. In the model, there are two types of costs: one is that parents must buy some final goods to raise their children and the other is that parents must sacrifice their time for raising their children. We further introduce survival probability of older individuals. This makes it possible to examine how longer life expectancy affects economic growth. Using this setting, we investigate two child-rearing policies: child allowance and child-care service. The child allowance means that a government gives a subsidy to an individual for each child she has. The childcare service means that the government employs people to provide childcare services. Therefore, this service can help parents save the time costs of raising children. Both policies positively impact the number of children; however, they raise the tax burden on individuals to finance their costs. This implies negative effects on the number of children. Consequently, those policies positively and negatively impact R&D activities. We investigate these effects in our analyses.

Related studies are those by Hirazawa and Yakita (2009), Strulik et al. (2013), Nakagawa et al. (2015), and Hashimoto and Tabata (2016). Hirazawa and Yakita (2009) examine a pay-as-you-go social security system by using an overlapping generations model for a small country. In their model, the number of children depends on private expenditure on final goods and the time spent by parents. However, they do not consider R&D activities and child-rearing policies. Strulik et al. (2013) consider a unified growth framework with micro-founded fertility and schooling behavior and construct a model that can explain historical facts. However, their methods mostly depend on numerical analyses, and they do not investigate any child-rearing policies. Nakagawa et al. (2015) use an overlapping generations model with human capital and examine possibilities of sustainable growth under a decreasing population size; however, they also do not consider R&D activities and any child-rearing policies. The study by Hashimoto and Tabata (2016) is closely related to our paper. They construct an overlapping generations model with R&D activities and child rearing costs as opportunity costs. Furthermore, they introduce survival probability of older individuals and examine the impact of rising longevity on economic growth. However, because their model does not have the child-rearing cost paid in final goods, the optimal fertility decision is constant for all time. Thus, they do not examine the relation between population dynamics and economic growth. We examine this relation in our setting.

The remainder of this paper is organized as follows. Section 2 establishes the model used in this study. Section 3 examines the equilibrium dynamics of the economy, threshold for the sustainability of economic growth, and effects of increasing life expectancy. Section 4 analyzes how child-rearing policies affect the sustainability of economic growth. Section 5 examines the growth rates. Section 6 investigates the model numerically. Finally, Section 7 concludes the paper.

2. Model

We consider a three-period overlapping-generations model following Diamond

(1965). An individual lives for three periods: childhood, young and old periods. In their childhood period, they do not make economic decisions. For simplicity, we assume that every individual certainly survives to the end of young period. In their young period, every individual supplies labor to the market and earns labor income. Furthermore, these young individuals raise their children. From a young to old period, an individual dies with a probability $1-p \in [0, 1)$. That is, the probability of an individual surviving to the next period is $p \in (0, 1]$. In the old period, they retire, consume their savings, and have no bequest motive. Let n_t be the number of children for each young individual. In each period, the population size of young individuals is given by N_t. As a result, the dynamics of a young individual population size, N_t, becomes

$$N_{t+1}=n_t N_t. \tag{1}$$

2.1 Individuals

Every young individual at period t maximizes the following utility:

$$u_t=\log c_t^y+p\beta \log c_{t+1}^o+\gamma \log n_t \tag{2}$$

where c_t^y is the consumption when young, c_{t+1}^o is the consumption when old, β is the subjective discount rate, and γ is the weight on utility derived from the number of children. We assume that each young individual has one unit of time endowment. Raising children requires the following two kinds of costs: *time* and *goods*. When each young individual raises n_t units of children, they have to incur δn_t units of final goods and ρn_t units of time. Let w_t be the wage rate. The disposable working income becomes $(1-\rho n_t)w_t$. Thus, the budget constraint for each young individual can be expressed as

$$c_t^y+s_t+\delta n_t=(1-\rho n_t)w_t, \quad \delta>0, \quad \rho \in (0, 1), \tag{3}$$

where s_t represents the savings during youth. The disposable working income $(1-\rho n_t)w_t$ is divided into consumption, savings, and the cost of raising children.

As in Yaari (1965) and Blanchard (1985), we assume that there are actuarially fair insurance companies and that the annuities market is perfectly competitive. At the end of the young period, each individual deposits his or her savings with an insurance company. The company invests them and repays the returns of the investment to the surviving insured old individuals. Owing to perfect competition, the rate of return on the annuities becomes $(1+r_{t+1})/p$, where r_{t+1} is the interest rate in period $t+1$. Thus, the consumption of the surviving old individuals becomes

$$c_{t+1}^o=\frac{1+r_{t+1}}{p}s_t. \tag{4}$$

Using (2), (3), and (4), we solve the intertemporal utility maximization problem as follows[1]：

$$s_t = \frac{p\beta w_t}{1+p\beta+\gamma}, \tag{5}$$

$$n_t = \frac{\gamma w_t}{(1+p\beta+\gamma)(\rho w_t+\delta)}. \tag{6}$$

To consider the optimal choice of n_t, we rearrange (6) as follows:

$$\frac{\gamma w_t}{(1+p\beta+\gamma)n_t} = \rho w_t + \delta.$$

The left- and right-hand sides represent the marginal benefit of raising children and the total marginal cost of raising children, respectively. This equation indicates the following relation:

$$n_t \gtreqless 1 \Longleftrightarrow \frac{\gamma w_t}{1+p\beta+\gamma} \gtreqless \rho w_t + \delta.$$

To exclude the case where the rearing time marginal cost always exceeds the marginal benefit of raising children evaluated at $n_t=1$, we impose the following assumption:

Assumption 1. $\dfrac{\gamma}{1+p\beta+\gamma} > \rho.$

Under this assumption, we can state the following lemma:

Lemma 1. *$n_t \gtreqless 1$ holds when $w_t \gtreqless \bar{w}$, where* $\bar{w} \equiv \dfrac{(1+p\beta+\gamma)\delta}{\gamma-(1+p\beta+\gamma)\rho}.$

(6) implies that an increase in the wage rate w_t raises the fertility rate n_t. When the wage rate is sufficiently high, the marginal benefit of raising children is relatively larger than the total marginal cost of doing so. In this case, the fertility rate exceeds 1, that is, $n_t > 1$ holds.

2.2 Final goods sector

Following Romer (1990) and Jones (1995), we consider three production sectors: a

[1] The optimal choices of s_t and n_t do not depend on the interest rate because we use a logarithmic utility function for simplicity.

final goods sector, an intermediate goods sector, and an R&D sector. We first describe the final goods sector.

We assume that the final goods market is perfectly competitive. The production technology of the final good is given by

$$Y_t = L_{Y,t}^{1-\alpha} \int_0^{A_t} x_{j,t}^{\alpha}\, dj, \quad 0<\alpha<1, \tag{7}$$

where Y_t, $L_{Y,t}$, A_t, and $x_{j,t}$ represent the output level, labor input, the variety of intermediate goods, and the input of the jth intermediate good, respectively. From profit maximization in competitive markets, factor prices become equal to the marginal products:

$$w_t = (1-\alpha) L_{Y,t}^{-\alpha} \int_0^{A_t} x_{j,t}^{\alpha}\, dj = (1-\alpha)\frac{Y_t}{L_{Y,t}}, \tag{8}$$

$$q_{j,t} = \alpha L_{Y,t}^{1-\alpha} x_{j,t}^{\alpha-1}, \tag{9}$$

where $q_{j,t}$ is the price of the jth intermediate good. In this study, we normalize the price of final goods to one. From (9), the demand function for intermediate good j is as follows:

$$x_{j,t} = \left(\frac{\alpha}{q_{j,t}}\right)^{\frac{1}{1-\alpha}} L_{Y,t}. \tag{10}$$

2.3 Intermediate goods sector

We assume that each differentiated intermediate good is produced by a single firm because the intermediate good is infinitely protected by a patent. That is, the intermediate goods market is monopolistically competitive. We further assume that one unit of labor input produces one unit of a differentiated intermediate good. Therefore, the firm manufacturing intermediate good j (firm j) maximizes its own profit, $\pi_{j,t} = q_{j,t} x_{j,t} - w_t x_{j,t}$, subject to the demand function (10). Solving this maximization problem, we obtain the following price charged by firm j:

$$q_{j,t} = q_t = \frac{1}{\alpha} w_t. \tag{11}$$

Thus, all intermediate goods are priced equally. From (10) and (11), the output and monopoly profits of firm j become

$$x_{j,t} = x_t = \left(\frac{\alpha^2}{w_t}\right)^{\frac{1}{1-\alpha}} L_{Y,t}, \tag{12}$$

$$\pi_{j,t} = \pi_t = \frac{1-\alpha}{\alpha} w_t x_t. \tag{13}$$

2.4 R&D sector

Next, we consider the technology involved in developing a new intermediate good. Before entering the intermediate goods market, an entrepreneur must undertake R&D activity to develop a new blueprint. R&D activities require labor input and the R&D sector is perfectly competitive. We assume the following R&D technologies:

$$A_{t+1}-A_t=\theta_t L_{A,t}, \tag{14}$$

where θ_t and $L_{A\,t}$ represent the productivity of R&D and the amount of labor devoted to R&D. $A_{t+1}-A_t$ measures new intermediate goods. Following Romer (1990), Grossman and Helpman (1991), and Jones (1995), the productivity of R&D depends on current knowledge that is a result of previous R&D activities. We assume the following productivity of R&D:

$$\theta_t=\bar{\theta}A_t^{\phi}. \tag{15}$$

Thus, A_t also represents the stock of knowledge. As discussed by Jones (1995), we consider the parameter range where $0<\phi<1$. If an entrepreneur can invent a new blueprint, he/she acquires v_t, where v_t denotes the market values of new blueprints. The profit of entrepreneurs is given by $\pi_t^A=v_t(A_{t+1}-A_t)-w_tL_{A,t}=(v_t\theta_t-w_t)L_{A,t}$. Perfect competition prevails in R&D activities. Consequently, when R&D is undertaken, the following equality holds:

$$v_t=\frac{w_t}{\bar{\theta}A_t^{\phi}}. \tag{16}$$

On the other hand, when $v_t\theta_t<w_t$, R&D is not conducted because $\pi_t^A<0$.

Next, we consider the no-arbitrage condition. The shareholders of stocks earn dividends π_{t+1} and capital gains or losses $v_{t+1}-v_t$. Therefore, we obtain the following no-arbitrage condition: $r_{t+1}\,v_t=\pi_{t+1}+v_{t+1}-v_t$.

2.5 Market clearing condition

Labor is used for the production of final and intermediate goods and in R&D activities. As mentioned above, the labor supply of each individual is $1-\rho n_t$. Because the population size of young individuals at period t is N_t, the labor market clearing condition becomes

$$L_{Y,t}+A_t\,x_t+L_{A,t}=(1-\rho n_t)N_t. \tag{17}$$

The total saving of young individuals in period t must be used for investment in R&D

or for the purchase of existing stocks. Hence, the asset market clearing condition is given by

$$w_t L_{A,t} + A_t v_t = s_t N_t. \tag{18}$$

Finally, we consider the equilibrium condition of the final goods market. The final goods are used for consumption and for raising children. The final goods clearing condition is as follows: $Y_t = c_t^y N_t + c_t^o p N_{t-1} + \delta n_t N_t$. Note that because the survival probability is p, the population size of old individuals at period t becomes pN_{t-1}.

3. Equilibrium

3.1 Dynamics

We now show equilibrium paths in this economy. Substituting (12) into (7) yields

$$Y_t = \left(\frac{\alpha^2}{w_t}\right)^{\frac{\alpha}{1-\alpha}} A_t L_{Y,t}. \tag{19}$$

Using (8) and (19), we obtain

$$w_t = \hat{\alpha} A_t^{1-\alpha}, \tag{20}$$

where $\hat{\alpha} \equiv \alpha^{2\alpha}(1-\alpha)^{1-\alpha}$. This implies that an increase in A_t raises the wage rate w_t. From (6) and (20), we obtain

$$n_t = \frac{\gamma \hat{\alpha} A_t^{1-\alpha}}{(1+p\beta+\gamma)(\rho \hat{\alpha} A_t^{1-\alpha} + \delta)}. \tag{21}$$

(5), (16), and (18) yield

$$L_{A,t} = \frac{s_t N_t}{w_t} - \frac{A_t v_t}{w_t} = \frac{p\beta N_t}{1+p\beta+\gamma} - \frac{1}{\bar{\theta}} A_t^{1-\phi}. \tag{22}$$

The investment in R&D is obtained from a difference between total savings and the purchase of existing stocks. Using (14), (15), and (22), we obtain

$$A_{t+1} = \frac{p\beta\bar{\theta}}{1+p\beta+\gamma} A_t^{\phi} N_t. \tag{23}$$

Equations (1), (21), and (23) characterize the dynamic system with respect to A_t and N_t. Note that both A_t and N_t in period t are predetermined variables.

3.2 Initial conditions for sustainable growth

First, we examine whether $N_{t+1} \geq N_t$ or $N_{t+1} < N_t$ at each point on the (A_t, N_t) plane. (1) implies that $N_{t+1} \gtreqless N_t$ holds when $n_t \gtreqless 1$. From (20) and Lemma 1, we obtain

$$n_t \gtreqless 1 \Longleftrightarrow A_t \gtreqless \left\{ \frac{\delta(1+p\beta+\gamma)}{\bar{\alpha}[\gamma-(1+p\beta+\gamma)\rho]} \right\}^{\frac{1}{1-\alpha}} \equiv \bar{A}.$$

Therefore, we have $N_{t+1} > N_t$ when $A_t > \bar{A}$ and $N_{t+1} < N_t$ when $A_t < \bar{A}$ [2].

We then investigate whether $A_{t+1} > A_t$ or $A_{t+1} = A_t$ at each point on the (A_t, N_t) plane. Because there is no product obsolescence, $A_{t+1} < A_t$ does not hold in this study. Setting $A_{t+1} = A_t$ in (23) leads to

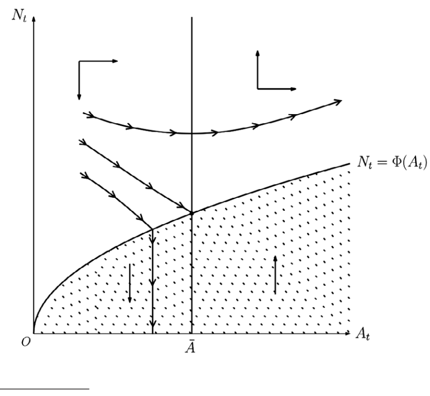
$$N_t = \frac{1+p\beta+\gamma}{p\beta\bar{\theta}} A_t^{1-\phi} \equiv \Phi(A_t). \tag{24}$$

Hence, we have $A_{t+1} > A_t$ above this locus and $A_{t+1} = A_t$ below this locus (hereafter, the no R&D region). Using these results, we can present a phase diagram as in Figure 1.

As shown in Figure 1, the point $(\bar{A}, \bar{N})$ is saddle-point stable, where $\bar{N} \equiv \Phi(\bar{A})$ [3]. Note that A_t and N_t are predetermined variables at time t. Hence, the initial state of the economy is given by a point (A_0, N_0) on the (A_t, N_t) plane. For an economy where the initial state (A_0, N_0) is in the upper-right side of the saddle-point stable arm, A_t increases and N_t decreases at first. After exceeding the vertical line $\bar{A}$, both A_t and N_t increase forever, that is, economic growth is sustainable. In contrast, for an economy where the initial state (A_0, N_0) is in the lower-left side of the saddle-point stable arm, the economy reaches the no R&D region in a finite time. After that time, A_t remains constant and N_t decreases at a constant rate. That is, the population decreases and R&D activities are not conducted. This path falls into the trap in the long run.

These results show that the saddle-point stable arm in Figure 1 represents the threshold of A_t for each level of N_t to sustain economic growth. In summary, we can

Figure 1 : Phase diagram of the dynamics of A_t and N_t.



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[2] Under Assumption 1, we can show $\bar{A} > 0$.

[3] In Appendix A, we show that the point $(\bar{A}, \bar{N})$ is locally saddle-point stable.

state the following proposition:

Proposition 1. *There is a threshold curve to sustain economic growth. The threshold curve is downward sloping on the* (A_t, N_t) *plane.*

From (20), the wage rate, w_t, is determined by the stock of knowledge A_t. Thus, a sufficiently small A_t implies that w_t is also low. From Lemma 1, the fertility rate n_t tends to be lower than 1. This induces the economy to fall into the trap. However, if the population size, N_t, is sufficiently large, the total savings become large. From (22), this expands investment in R&D, and hence, the growth rate of A_t increases. The economy can exceed the threshold value $\bar{A}$ and attain sustainable growth.

3.3 Effects of a rise in life expectancy

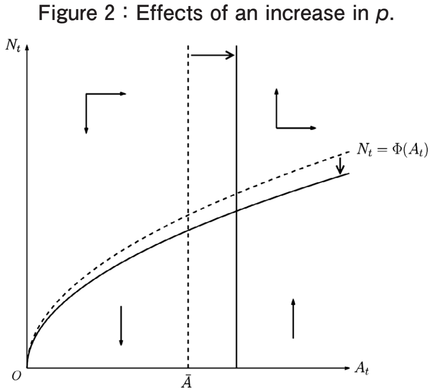
We examine the effects of a rise in life expectancy (that is, an increase in p) on the sustainability of economic growth. Differentiating $\bar{A}$ with respect to p yields

$$\frac{\partial \bar{A}}{\partial p}=\frac{1}{1-\alpha}\left\{\frac{\delta(1+p\beta+\gamma)}{\tilde{\alpha}[\gamma-(1+p\beta+\gamma)\rho]}\right\}^{\frac{\alpha}{1-\alpha}}\frac{\delta\gamma\beta}{\tilde{\alpha}[\gamma-(1+p\beta+\gamma)\rho]^2}>0.$$

Moreover, from (24), we obtain $\partial\Phi(A_t)/\partial p<0$. A rise in p shifts the $A_t=\bar{A}$ locus rightward and the $N_t=\Phi(A_t)$ locus downward (see Figure 2). These results are summarized in the following proposition:

Proposition 2. *An increase in life expectancy* p *enlarges the region where R&D is conducted, whereas it shrinks the region where population increases.*

Figure 2: Effects of an increase in p.



From (5) and (6), a higher p implies a higher per capita saving s_t and a lower fertility rate n_t. Because individuals survive with a higher probability, they decrease the number of children and increase the savings for consumption in their old age. From (22), we see that increasing per capita saving accelerates the investment in R&D. From these results, the R&D region expands and the region where population increases shrinks. The sustainability of economic growth depends on these opposite effects. We will see the effects of p on the sustainability of economic growth by numerical examples in Section 6.

4. Child-rearing policies

As discussed in the previous section, an increase in life expectancy negatively impacts the fertility rate. Therefore, we analyze the effects of child rearing policies aimed at increasing the fertility rate[4]. First, we introduce a child allowance. Suppose the government subsidizes a fraction of $z \in [0, 1]$ of the rearing cost paid in the final goods. Second, we introduce a childcare service as follows. We assume that the rearing cost spent by young individual's time ρ depends on public goods B_t. One unit of a public good requires one unit of labor input. ρ is a function of per capita B_t; that is, $\rho(b_t)$, where $b_t \equiv B_t/N_t$. In addition, $\rho(b_t)$ is assumed to satisfy the following properties: $\rho'(b) < 0$ and $\rho''(b) > 0$. The government expenditure is financed by a lump-sum tax τ_t on each young individual. In each period, the government runs a balanced budget. Thus, the government's budget constraint in period t can be expressed as

$$\tau_t = z\delta n_t + w_t b_t. \tag{25}$$

For simplicity, we assume that the government keeps b_t constant (that is, $b_t = b$) and that τ_t is determined to satisfy the government budget constraint.

Under these policies, we rewrite the budget constraint of each young individual as follows: $c_t^y + s_t + (1-z)\delta n_t = [1-\rho(b_t)n_t]w_t - \tau_t$. Individuals' optimal decisions (5) and (6) become

$$s_t = \frac{p\beta(w_t - \tau_t)}{1 + p\beta + \gamma}, \tag{26}$$

$$n_t = \frac{\gamma(w_t - \tau_t)}{(1 + p\beta + \gamma)[\rho(b)w_t + (1-z)\delta]}. \tag{27}$$

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[4] Many developed countries implement various policies to improve the child-rearing environment. For example, in Japan, the government has been increasing child allowances and aiming at improving childcare services. Therefore, in this study, we consider the effects of these policies.

Using (20), (25), (26), and (27), we obtain

$$s_t=\frac{(1-b)p\beta[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]}{(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}+[(1-z)(1+p\beta)+\gamma]\delta}. \tag{28}$$

$$n_t=\frac{(1-b)\gamma\tilde{\alpha}A_t^{1-\alpha}}{(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}+[(1-z)(1+p\beta)+\gamma]\delta}. \tag{29}$$

(16), (18), and (28) yield

$$L_{A,t}=\frac{(1-b)p\beta[(1-z)\delta+p(b)\tilde{\alpha}A_t^{1-\alpha}]N_t}{(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}+[(1-z)(1+p\beta)+\gamma]\delta}-\frac{1}{\bar{\theta}}A_t^{1-\phi}. \tag{30}$$

As mentioned above, the investment in R&D is obtained from the difference between total savings and the purchase of existing stocks even under child rearing policies. From (14), (15), and (30), we obtain

$$A_{t+1}=\frac{(1-b)p\beta[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]\bar{\theta}A_t^{\phi}N_t}{(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}+[(1-z)(1+p\beta)+\gamma]\theta}. \tag{31}$$

Equations (1), (29), and (31) characterize the dynamic system with respect to A_t and N_t.

We investigate the phase diagram under child-rearing policies. Using (29), we obtain

$$n_t\gtreqless 1 \Leftrightarrow A_t\gtreqless\left\{\frac{\delta[(1-z)(1+p\beta)+\gamma]}{\tilde{\alpha}[(1-b)\gamma-(1+p\beta+\gamma)\rho(b)]}\right\}^{\frac{1}{1-\alpha}}\equiv\hat{A}. \tag{32}$$

Therefore, we have $N_{t+1}\gtreqless N_t$ when $A_t\gtreqless\hat{A}$. To ensure the existence of $\hat{A}$, we impose the following assumption (see Appendix B for details)[5] :

Assumption 2. $\gamma>(1+p\beta+\gamma)\rho(0)$ *and* $0\le b<\hat{b}$, *where* $\hat{b}$ *is defined as* $(1-\hat{b})\gamma-(1+p\beta+\gamma)\rho(\hat{b})=0$.

Next, we examine whether $A_{t+1}>A_t$ or $A_{t+1}=A_t$ at each point on the (A_t, N_t) plane. Setting $A_{t+1}=A_t$ in (31) leads to

$$N_t=\frac{(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}+[(1-z)(1+p\beta)+\gamma]\delta}{p\beta\bar{\theta}(1-b)[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]}A_t^{1-\phi}\equiv\Gamma(A_t).$$

We have $A_{t+1}>A_t$ when $N_t>\Gamma(A_t)$ and $A_{t+1}=A_t$ when $N_t\le\Gamma(A_t)$. we show in Appendix C that $N_t=\Gamma(A_t)$ locus is increasing in A_t on the (A_t, N_t) plane und0er the following assumption:

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5 Assumption 2 represents $(1-b)\gamma-(1+p\beta+\gamma)\rho(b)>0$. From (29), if $(1-b)\gamma-(1+p\beta+r)\rho(b)\le 0$, we can show that $n_t<1$ for all A_t. Hence, Assumption 2 corresponds to Assumption 1.

Assumption 3. $z \leq \min\{\hat{z}, 1\}$, *where* $\hat{z}$ *is defined in Appendix C.*

Under these two assumptions, we can conclude that the phase diagram of child-rearing policies is similar to that of the laissez-faire economy.

Effects of child allowance

We begin with the effects of changes in z on the sustainability of economic growth. From (32), we obtain $\partial\widehat{A}/\partial z<0$, and thus, a rise in z shifts the $A_t=\widehat{A}$ locus leftward. Differentiating $\Gamma(A_t)$ with respect to z, we obtain

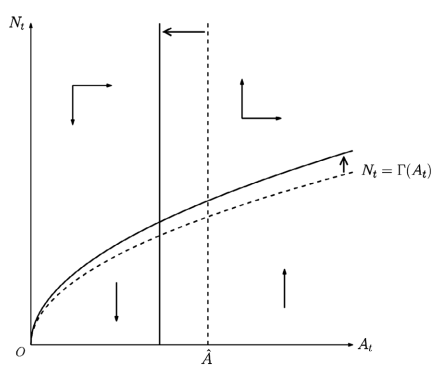
$$\frac{\partial\Gamma(A_t)}{\partial z}=\frac{\gamma\delta A_t^{1-\phi}}{p\beta\bar{\theta}(1-b)}\frac{\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}}{[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]^2}>0.$$

Therefore, when the government increases z, the locus of $N_t=\Gamma(A_t)$ shifts upward. We can depict these shifts in Figure 3 and state the following proposition:

Proposition 3. *A rise in the child allowance z reduces the region where R&D is conducted, whereas it enlarges the region where population increases.*

The effects of child allowance is opposite to those of the increase in life expectancy. Subsidizing the rearing cost positively impacts the fertility rate. However, from (25) and (26), a higher z leads to a higher tax burden on young individuals. This negative effect reduces per capita savings, and decreases investment in R&D. From these results, when the government raises z, the R&D region shrinks and the region where population increases expands. We also consider the effects of z on the sustainability of economic growth through numerical examples in Section 6.

Figure 3：Effects of an increase in z or b when $0\leq b<\tilde{b}$.



Effects of child care service

Next, we examine the effects of changes in b. Differentiating $\hat{A}$ with respect to b yields

$$\frac{\partial \hat{A}}{\partial b}=\frac{[\gamma+(1+p\beta+\gamma)\rho'(b)]\hat{A}}{(1-\alpha)[(1-b)\gamma-(1+p\beta+\gamma)\rho(b)]}. \tag{33}$$

From the existence condition of $\hat{A}$, we obtain $(1-b)\gamma-(1+p\beta+\gamma)\rho(b)>0$ if $0\leq b<\hat{b}$. Thus, the sign of $\partial\hat{A}/\partial b$ is determined by the sign of the term $\gamma+(1+p\beta+\gamma)\rho'(b)$. Here, we impose the following assumption:

Assumption 4. $\gamma+(1-p\beta+\gamma)\rho'(0)<0$.

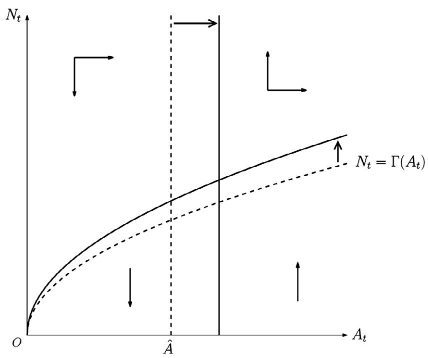
Under Assumption 4, we can show in Appendix D that $\tilde{b}<\hat{b}$ and $\partial\hat{A}/\partial b \lesseqgtr 0$ if $b \lesseqgtr \tilde{b}$, where $\tilde{b}$ is defined as $\gamma+(1+p\beta+r)\rho'(\tilde{b})=0$. We then differentiate $\Gamma(A_t)$ with respect to b as follows:

$$\frac{\partial\Gamma(A_t)}{\partial b}=\frac{A_t^{1-\phi}}{p\beta\theta}\left\{\frac{[(1-z)(1+p\beta)+\gamma]\delta+(1+p\beta+\gamma)\rho(b)\tilde{\alpha}A_t^{1-\alpha}}{(1-b)^2[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]}\right.$$
$$\left.-\frac{z\gamma\delta\rho'(b)\tilde{\alpha}A_t^{1-\alpha}}{(1-b)[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]^2}\right\}>0.$$

We summarize these results as follows. If $0\leq b<\tilde{b}$, an increase in b shifts the $A_t=\hat{A}$ locus leftward and the $N_t=\Gamma(A_t)$ locus upward (see Figure 3). On the other hand, if $\tilde{b}<b<\hat{b}$, a rise in b shifts the $A_t=\hat{A}$ locus rightward and the $N_t=\Gamma(A_t)$ locus upward (see Figure 4). Thus, we have the following proposition:

Proposition 4. *When $0\leq b<\tilde{b}$, an increase in childcare service b decreases the region*

Figure 4 : Effects of an increase in b when $\tilde{b}\leq b<\hat{b}$.



where R&D is conducted, whereas it expands the region where population increases. When $\tilde{b}<b<\hat{b}$, an increase in childcare service reduces both regions where R&D is conducted and where population increases.

To understand the intuition of Proposition 4, we consider individuals' optimal decision. Using (25) and taking the total differentials of (27) yields

$$\left[-\rho'(b)n_t-\frac{\gamma}{1-p\beta+\gamma}\right]w_t db=\left[\rho(b)w_t+(1-z)\delta+\frac{\gamma z\delta}{1-p\beta+\gamma}\right]dn_t.$$

The first term of the left-hand side represents the marginal benefit through the reduction in the rearing time. Since expanding childcare service decreases the rearing time, each young individual can supply more labor and earn more disposable working income. The second term of the left-hand side represents the marginal cost through the tax burden of childcare service. If the former effect is larger (smaller) than the latter effect, $dn_t/db>0(dn_t/db<0)$ holds[6]. As well as the effect of z, a rise in b decreases per capita savings because of the tax burden effect. Therefore, if $0\leq b<\tilde{b}$, the effect of b is similar to that of z. When the government increases b, the R&D region shrinks and the region where population increases expands. In contrast, if $\tilde{b}<b<\hat{b}$, the tax burden effect is very large. An increase in b decreases both regions where R&D is conducted and where population increases. In this case, expanding childcare service is not appropriate for the sustainability of economic growth. We will see the effects of b on the sustainability of economic growth through the numerical examples presented in Section 6.

5. Growth rates

Thus far, we have focused on the sustainability of economic growth. We must consider the growth rate of per capita GDP. As in Strulik et al. (2013) and Hashimoto and Tabata (2016), we define per capita GDP as $y_t\equiv Y_t/N_t$. Substituting (20) into (19), we obtain $y_t=[\alpha^2/(1-\alpha)]^{\alpha}A_t^{1-\alpha}l_{Y,t}$, where $l_{Y,t}\equiv L_{Y,t}/N_t$. The growth rate of per capita GDP is as follows:

$$1+g_{y,t}\equiv\frac{y_{t+1}}{y_t}=\left(\frac{A_{t+1}}{A_t}\right)^{1-\alpha}\frac{l_{Y,t+1}}{l_{Y,t}} \tag{34}$$

[6] Note that Proposition 4 is evaluated at $n_t=1$.

The growth rate of per capita GDP is determined by the growth of A_t and by the changes in $l_{Y,t}$. Let us define the growth rate of A_t as $1+g_{A,t}\equiv A_{t+1}/A_t$. If $L_{A,t}>0$, we can calculate $g_{A,t}$ using (31). We then consider the labor market clearing condition to derive $l_{Y,t}$. Using (12) and (20), the labor input into the production of intermediate goods becomes $A_t x_t=\alpha^2 L_{Y,t}/(1-\alpha)$. From (30), when R&D is conducted, $L_{A,t}>0$ holds. Under child-rearing policies, (17) can be expressed as $L_{Y,t}+A_t x_t+L_{A,t}=[1-b-\rho(b)n_t]N_t$. Hence, these results imply

$$l_{Y,t}=\frac{1-\alpha}{1-\alpha+\alpha^2}[1-b-\rho(b)n_t-l_{A,t}] \ \ if \ \ L_{A,t}>0, \tag{35}$$

where $l_{A,t}\equiv L_{A,t}/N_t$. Using (29), (30), (31), (34), and (35), we can calculate the growth rate of per capita GDP $g_{y,t}$. Furthermore, we can show $g_{y,t}=0$ if $L_{A,t}=0$[7].

If the economy is on a sustainable growth path, A_t and N_t grow forever. We define the balanced growth path (BGP) as the state of the economy at which the growth rates and the employment share of each sector become constant. Using (29), n_t along the BGP is given by

$$\lim_{A_t\to\infty} n_t=\frac{(1-b)\gamma}{(1+p\beta+\gamma)\rho(b)}\equiv n^*. \tag{36}$$

In this study, asterisks represent variables along the BGP. As shown in Appendix E, we can derive the growth rates along the BGP as follows:

$$1+g_A^*=(n^*)^{\frac{1}{1-\phi}},$$
$$1+g_y^*=(n^*)^{\frac{1-\alpha}{1-\phi}}.$$

As in Jones (1995), the growth rates along the BGP are determined by the fertility rate. We then consider the effects of changes in p, z, and b on n^*. From (36), a rise in p reduces n^* and z does not affect n^*. As shown in Appendix F, we obtain

$$\frac{\partial n^*}{\partial b}\gtreqless 0 \Leftrightarrow b\lesseqgtr \bar{b}, \tag{37}$$

where $\bar{b}$ is defined as $\rho(\bar{b})+(1-\bar{b})\rho'(\bar{b})=0$. Therefore, there exists $\bar{b}$ that maximizes the long-run growth rate. However, does the economy achieve sustainable growth under this value? How long does it take to attain this maximal rate? We will see these answers in the following numerical analyses.

At the end of this section, we mention why z has no effect on n^*. As discussed above, when the stock of knowledge A_t is larger, the wage rate w_t is higher. If the economy

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7 (29) and $L_{A,t}=0$ imply that $A_{t+1}=A_t$ and n_t becomes constant (that is, $n_{t+1}=n_t$). Hence, from (35), $l_{Y,t+1}=l_{Y,t}$ holds if $L_{A,t}=0$. As a result, by using (34), we obtain $g_{y,t}=0$ when $L_{A,t}=0$.

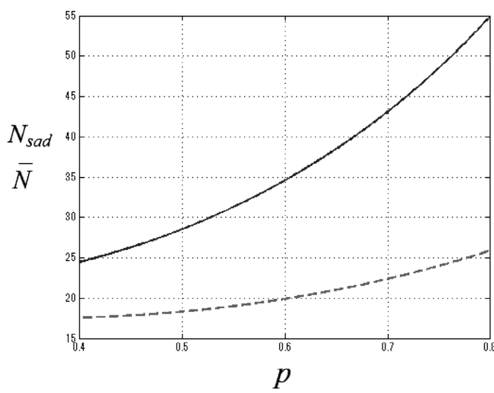
achieves sustainable growth, the wage rate continues to rise. Thus, the rearing cost paid in final goods, $(1-z)\delta n_t$, becomes negligibly low and the individual's fertility decision depends on the rearing time cost in the long run[8].

6. Numerical results

We analyze the model numerically because investigating the aforementioned effects of changes in p, z, and b on the sustainability of economic growth and the growth rates along the transitional dynamics is too complicated. First, we consider the effects of an increase in life expectancy as discussed in Subsection 3.3. In the dynamic system characterized by equations (1), (21), and (23), we adopt the following parameters: $\beta=1, \gamma=0.3, \alpha=0.8, \delta=0.1, \phi=0.7, \bar{\theta}=1$, and $\rho=0.1$. Under $A_0=20$, we calculate the initial value of population size that converges to the stable saddle-point $(\bar{A}, \bar{N})$. Let us define N_{sad} as this initial value. The solid and dashed lines in Figure 5 present N_{sad} and $\bar{N}$ for each value of $p\in[0.4, 0.8]$. From Figure 5 and Proposition 2, when p increases, the point $(\bar{A}, \bar{N})$ shifts up and rightward and N_{sad} rises. This shows that the saddle-point stable arm shifts up and rightward, that is, the region of sustainable growth shrinks. As mentioned in Subsection 3.3, a0n increase in p raises the R&D region and reduces the region where population increases. As shown in Figure 5, the latter exceeds the former effect.

Next, we set $A_0=20$ and $N_0=40$ and vary the value of p from 0.4 to 0.8 at intervals

Figure 5 : Effects of changes in p on the sustainability of economic growth. The solid line is N_{sad}. The dashed line is $\bar{N}$.



[8] From (25) and (27), we can also confirm that $n_t=n^*$ holds for all time if $\delta=0$.

of 0.1. Figure 6 presents the evolution of n_t, $g_{A,t}$, and $g_{y,t}$ for each value of p. To see the short-run effects, we provide the short-run values of n_t, $g_{A,t}$, and $g_{y,t}$ in Table 1. Initially, population size N_t decreases, whereas the stock of knowledge A_t grows because $n_t<1$ and $g_{A,t}>0$ from Table 1. When $p=0.4$, $p=0.5$, and $p=0.6$, the economy attains sustainable growth because $n_t>1$ holds in a finite time. However, when $p=0.7$ and $p=0.8$, the economy falls into the trap (that is, $n_t<1$ and $g_{A,t}=0$ hold in a finite time). This result is consistent with that of Figure 5. As mentioned in Subsection 3.3, an increase in p raises per capita saving s_t and decreases the fertility rate n_t. Table 1 shows that investment in R&D initially expands but starts to decrease after a few periods. From (22), as the population size falls, the total savings decrease, and this reduces investment in R&D. Table 1 also shows that the initial increase in R&D

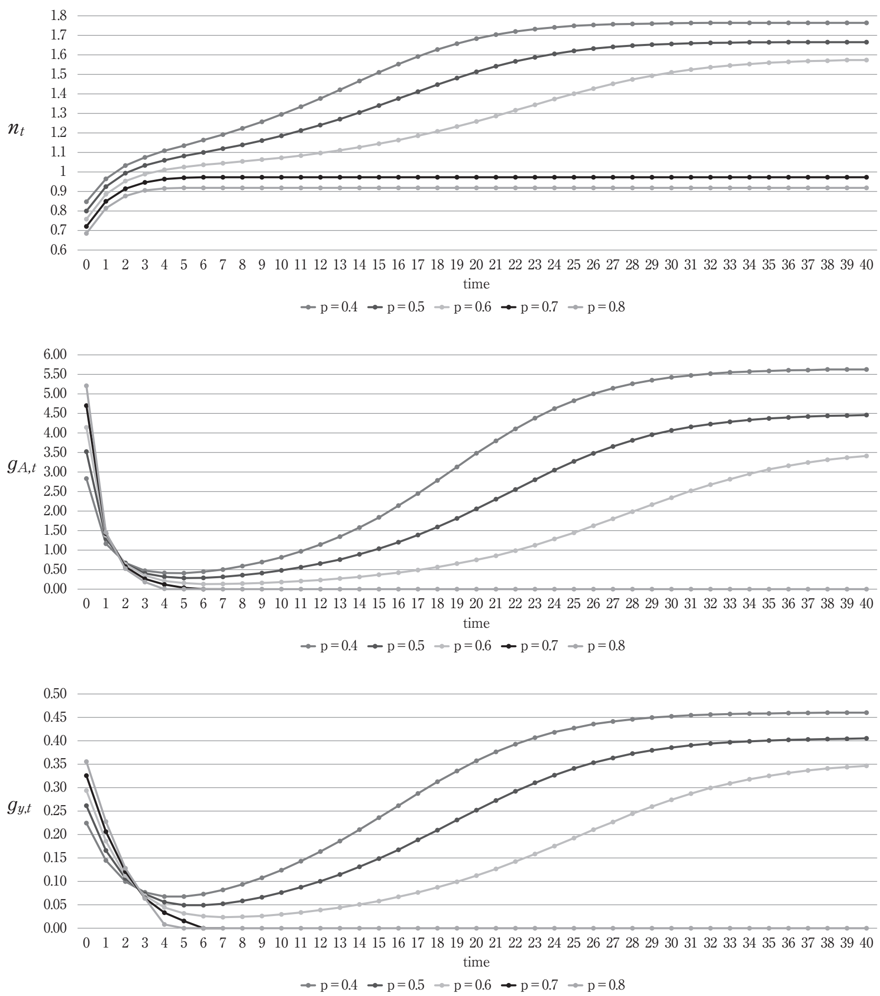
Table 1 : The short-run values of n_t, $g_{A,t}$, and $g_{y,t}$ in Figure 6.

			n_t		
t	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$
0	0.847	0.800	0.758	0.720	0.686
1	0.965	0.925	0.887	0.850	0.815
2	1.033	0.993	0.953	0.915	0.878
3	1.076	1.033	0.990	0.947	0.906
4	1.108	1.061	1.011	0.963	0.917
5	1.136	1.082	1.026	0.971	0.918
6	1.163	1.101	1.036	0.973	0.918

			$g_{A,t}$		
t	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$
0	2.831	3.523	4.142	4.699	5.203
1	1.169	1.301	1.385	1.435	1.461
2	0.660	0.659	0.629	0.585	0.532
3	0.472	0.415	0.342	0.262	0.183
4	0.410	0.318	0.216	0.115	0.019
5	0.409	0.287	0.160	0.040	0.000
6	0.444	0.290	0.138	0.000	0.000

			$g_{y,t}$		
t	$p=0.4$	$p=0.5$	$p=0.6$	$p=0.7$	$p=0.8$
0	0.223	0.260	0.293	0.324	0.355
1	0.145	0.165	0.185	0.206	0.227
2	0.099	0.106	0.112	0.119	0.128
3	0.076	0.072	0.068	0.065	0.064
4	0.067	0.056	0.044	0.033	0.008
5	0.067	0.049	0.031	0.015	0.000
6	0.073	0.049	0.025	0.000	0.000

Figure 6 : Numerical example (life expectancy rising).



investment is not large, and as a result, an economy of higher p can be trapped in the no R&D region. With regard to the growth rate of per capita GDP $g_{y,t}$ Figure 6 shows that the path of $g_{y,t}$ is similar to that of $g_{A,t}$. This implies that the effect of changes in $l_{Y,t}$ on $g_{y,t}$ is sufficiently small from (34). As in Section 5, Figure 6 shows that a higher p leads to lower fertility and growth rates in the long run.

Child-rearing policies

To examine the effects of child-rearing policies, we specify the functional form of rearing time cost $\rho(b)$ as follows:

$$\rho(b)=\frac{\lambda}{1+\Delta b^{\psi}}.$$

We choose the parameter values as $\lambda=0.1$, $\Delta=2.5$, and $\psi=0.8$. For other parameters, we adopt $\beta=1$, $\gamma=0.3$, $\alpha=0.8$, $\delta=0.1$, $\phi=0.7$, $\bar{\theta}=1$, and $p=0.7$ in the dynamic system characterized by equations (1), (29) and (31). Under this parameter set, we obtain $\tilde{b}=0.113$, $\hat{b}=0.782$, and $\bar{b}=0.273$.

First, we investigate the effects of child allowance on the sustainability of economic growth. Under $b=0$ and $A_0=20$, we calculate N_{sad}. The solid and dashed lines in Figure 7 present N_{sad} and $\widehat{N}$ for each value of $z\in[0, 0.2]$. From Figure 7 and Proposition 3, when the government raises z, the point $(\widehat{A}, \widehat{N})$ shifts left and downward and N_{sad} decreases. This means that the saddle-point stable arm shifts left and downward, that is, the region of sustainable growth enlarges. As mentioned in Section 4, an increase in z reduces the R&D region and enlarges the region where population increases. As shown in Figure 7, the latter exceeds the former effect.

We then set $b=0$, $A_0=20$, and $N_0=40$. Furthermore, we vary the value of z from 0 to 0.2 at intervals of 0.05. Note that the case of $z=0$ is equivalent to the case of $p=0.7$ in Figure 6; that is, the economy falls into the trap. Figure 8 and Table 2 present the evolution and short-run values of the same variables in Figure 6 and Table 1 for each value of z. Figure 8 shows that a higher z implies a higher n_t and that the economy can achieve the sustainable growth. As discussed in Section 4, an increase in z raises the tax burden on young individuals. This reduces per capita savings and investment in R&D. As shown in Table 2, although a higher z implies a lower $g_{A,t}$ initially, investment

Figure 7 : Effects of changes in z on the sustainability of economic growth. The solid line is N_{sad}. The dashed line is $\hat{N}$.

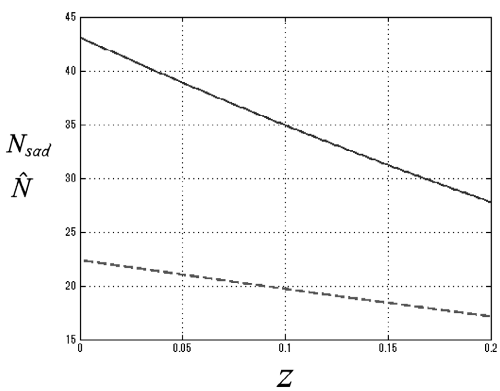
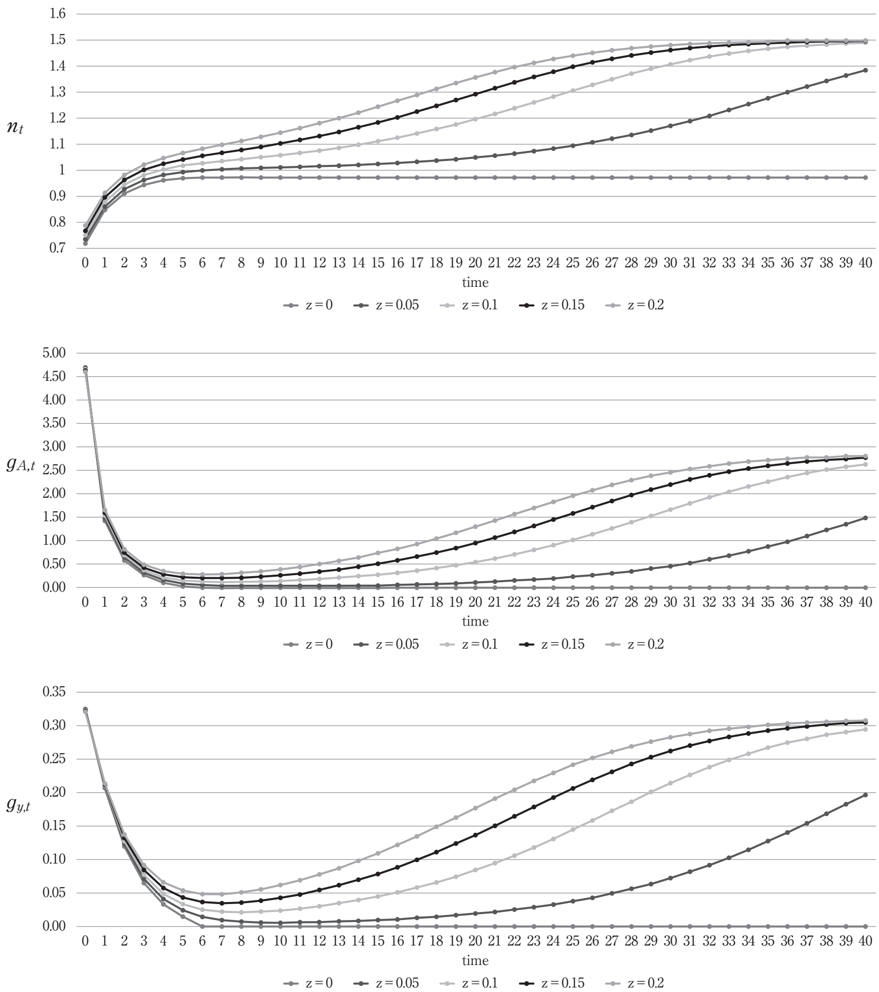


Figure 8：Numerical example（child allowance）.



in R&D expands in the next period. This is because the effect of fertility enhancement raises the total savings. Similar to the numerical analysis of p, $g_{y,t}$ depends on $g_{A,t}$. As mentioned in Section 5, we can confirm that each path converges to the same fertility rate if the economy is on a sustainable growth path.

Next, we consider the effects of child care service on the sustainability of economic growth. Under $z=0$ and $A_0=20$, we calculate N_{sad}. The solid and dashed lines in Figure 9 present N_{sad} and $\widehat{N}$ for each value of $b\in[0, 0.2]$. From Figure 9 and Proposition 4, when b is sufficiently small (large), an increase in b expands (reduces) the region of sustainable growth. As mentioned in Section 4, if b is sufficiently large, the

Table 2: The short-run values of n_t, $g_{A,t}$, and $g_{y,t}$ in Figure 8.

		n_t			
t	z=0	z=0.05	z=0.1	z=0.15	z=0.2
0	0.720	0.736	0.753	0.771	0.790
1	0.850	0.866	0.882	0.899	0.916
2	0.915	0.931	0.948	0.966	0.984
3	0.947	0.966	0.984	1.004	1.024
4	0.963	0.984	1.006	1.027	1.049
5	0.971	0.995	1.019	1.044	1.068
6	0.973	1.001	1.029	1.057	1.084

		$g_{A,t}$			
t	z=0	z=0.05	z=0.1	z=0.15	z=0.2
0	4.699	4.677	4.653	4.628	4.602
1	1.435	1.485	1.536	1.591	1.647
2	0.585	0.637	0.693	0.752	0.814
3	0.262	0.315	0.372	0.431	0.495
4	0.115	0.170	0.228	0.291	0.357
5	0.040	0.099	0.162	0.229	0.300
6	0.000	0.063	0.132	0.206	0.284

		$g_{y,t}$			
t	z=0	z=0.05	z=0.1	z=0.15	z=0.2
0	0.324	0.323	0.322	0.321	0.320
1	0.206	0.207	0.209	0.210	0.212
2	0.119	0.123	0.127	0.132	0.137
3	0.065	0.071	0.077	0.084	0.091
4	0.033	0.041	0.049	0.057	0.066
5	0.015	0.024	0.033	0.043	0.053
6	0.000	0.015	0.026	0.037	0.049

tax burden effect becomes sufficiently large. This negatively impacts the sustainability of economic growth. On the other hand, if b is sufficiently small, the effect of b is similar to that of z.

To examine the dynamic effects, we set $z=0$, $A_0=20$, and $N_0=40$. We vary the value of b from 0 to 0.2 at intervals of 0.05, and hence, we obtain $\rho(0)=0.1$, $\rho(0.05)=0.082$, $\rho(0.1)=0.072$, $\rho(0.15)=0.065$, and $\rho(0.2)=0.059$. As well as the analysis of child allowance, the case of $b=0$ is equivalent to the case of $p=0.7$ in Figure 6. Figure 10 and Table 3 show that an increase in b does not necessarily raise the fertility rate in the short run. From (37) and $\bar{b}=0.273$, the long-run fertility rate becomes higher when $b=0.2$. However, it takes a long time to achieve a higher fertility rate and

Figure 9：Effects of changes in b on the sustainability of economic growth.
The solid line is N_{sad}. The dashed line is $\hat{N}$.

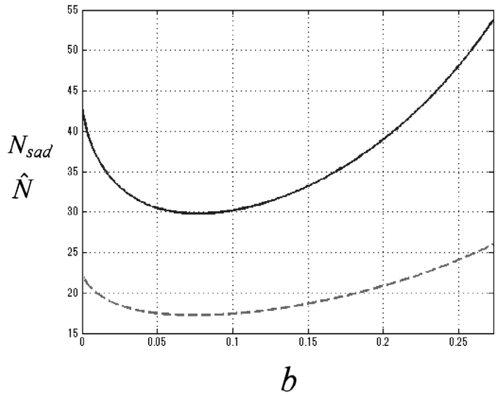


Figure 10：Numerical example (child care service).

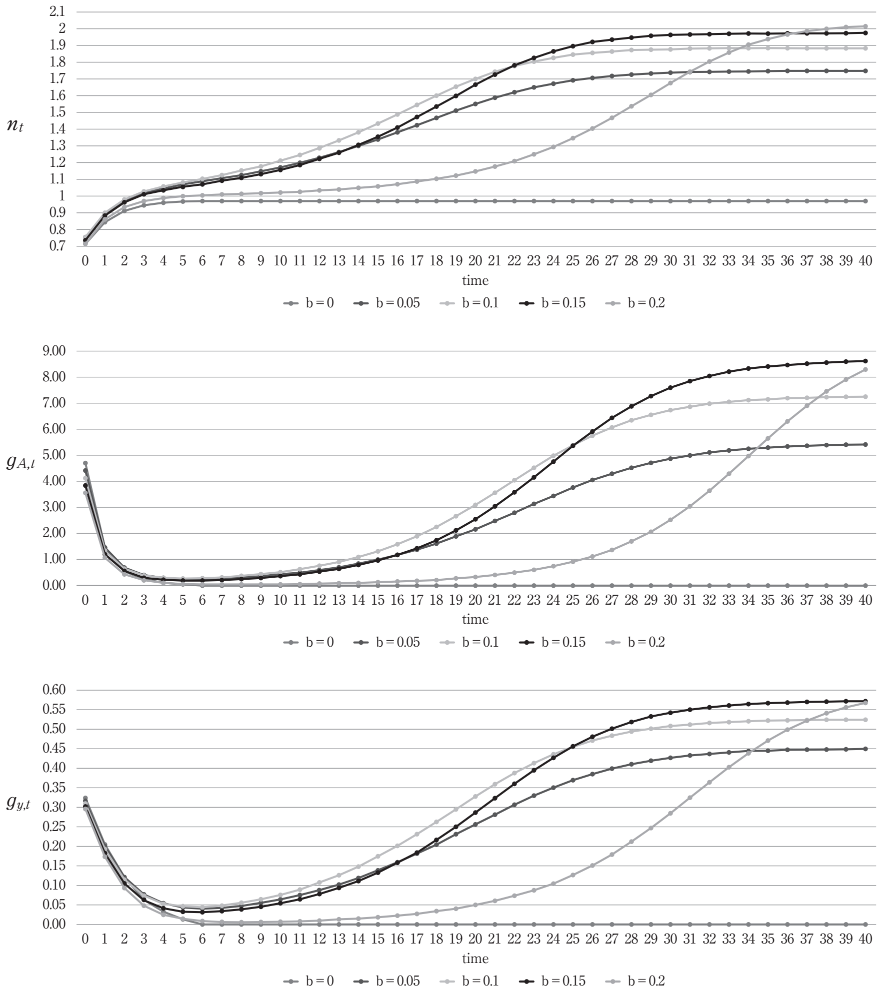


Table 3: The short-run values of n_t, $g_{A,t}$, and $g_{y,t}$ in Figure 10.

	n_t					
t	b=0	b=0.05	b=0.1	b=0.15	b=0.2	b=0.273
0	0.720	0.751	0.750	0.737	0.717	0.676
1	0.850	0.898	0.902	0.888	0.862	0.810
2	0.915	0.976	0.982	0.967	0.935	0.870
3	0.947	1.020	1.029	1.010	0.972	0.892
4	0.963	1.049	1.060	1.037	0.991	0.893
5	0.971	1.070	1.084	1.057	1.001	0.893
6	0.973	1.089	1.106	1.074	1.008	0.893

	$g_{A,t}$					
t	b=0	b=0.05	b=0.1	b=0.15	b=0.2	b=0.273
0	4.699	4.414	4.129	3.844	3.559	3.143
1	1.435	1.449	1.357	1.225	1.072	0.828
2	0.585	0.681	0.643	0.554	0.436	0.236
3	0.262	0.403	0.391	0.316	0.205	0.010
4	0.115	0.293	0.296	0.224	0.108	0.000
5	0.040	0.256	0.271	0.195	0.064	0.000
6	0.000	0.255	0.282	0.197	0.046	0.000

	$g_{y,t}$					
t	b=0	b=0.05	b=0.1	b=0.15	b=0.2	b=0.273
0	0.324	0.317	0.310	0.303	0.297	0.288
1	0.206	0.200	0.193	0.184	0.174	0.160
2	0.119	0.122	0.116	0.106	0.093	0.073
3	0.065	0.077	0.074	0.064	0.049	0.004
4	0.033	0.054	0.053	0.043	0.026	0.000
5	0.015	0.045	0.046	0.034	0.015	0.000
6	0.000	0.042	0.045	0.033	0.010	0.000

growth rates. In addition, under this parameter set and initial values of $A_0=20$ and $N_0=40$, if we adopt $b=\bar{b}$ to maximize the long-run growth rate, the economy falls into the trap in a finite time (see Table 3). These results show that we have to closely focus on the policy effects and condition of the economy while implementing this kind of policy.

7. Conclusion

This study constructs an overlapping generations model with R&D activities and endogenous fertility and investigates the sustainability of economic growth. We show that there is a trap region in which the economy cannot achieve sustainable growth.

This study also analyzes the effects of an increase in life expectancy. Furthermore, it shows that a rise in life expectancy expands the region where R&D is conducted, while it shrinks the region where population increases. Regarding the effects of child rearing policies, we show that under certain conditions, the effect of such policies is opposite to that of an increase in life expectancy.

This study presents several interesting directions for future research. First, we do not consider human capital investment. However, it is well known that human capital investment is one of the important factors of economic growth. In addition, most existing studies examine the trade-off between the number of children (quantity) and the level of education (quality). In future research, it would be interesting to incorporate human capital investment into the model. Second, welfare analyses can be conducted. For example, Futagami and Hori (2010) examine the socially optimal number of children by using an R&D-based growth model with endogenous fertility. One could also investigate welfare-improving child-rearing policies. Third, for simplicity, we assume that government expenditure is financed by a lump-sum tax on each young individual. Future work could consider the potentially interesting changes that would result from consumption or income taxes. Finally, we do not calibrate the model to Japan's data. Thus, investigating the current condition of the Japanese economy remains an area of future research.

Appendix

A. Local stability around the point $(\bar{A}, \bar{N})$

From (1), (21), and (23), approximating linearly around the point $(\bar{A}, \bar{N})$ yields

$$\begin{pmatrix} A_{t+1}-\bar{A} \\ N_{t+1}-\bar{N} \end{pmatrix} = \begin{pmatrix} J_{AA} & J_{AN} \\ J_{NA} & J_{NN} \end{pmatrix} \begin{pmatrix} A_t-\bar{A} \\ N_t-\bar{N} \end{pmatrix},$$

where

$$J_{AA}=\phi, \qquad J_{AN}=\frac{\bar{A}}{\bar{N}},$$

$$J_{NA}=\frac{(1-\alpha)\bar{N}}{A}\left[1-\frac{(1+p\beta+\gamma)\rho}{\gamma}\right], \quad J_{NN}=1.$$

Note that we use $(1+p\beta+\gamma)\bar{A}=p\beta\bar{\theta}\bar{A}^{\phi}\bar{N}$ and $\gamma\hat{\alpha}=(1+p\beta+\gamma)(\rho\hat{\alpha}+\delta\bar{A}^{\alpha-1})$ to derive $J_{ij}(i, j=A, N)$. Let us denote the two eigenvalues of the Jacobian matrix of the linearized system as ν_1 and ν_2. These eigenvalues are the roots of the characteristic polynomial $P(\nu)=\nu^2-trJ\nu+detJ$, where $trJ\equiv J_{AA}+J_{NN}$ and $detJ\equiv J_{AA}J_{NN}-J_{AN}J_{NA}$. Using J_{ij}, we obtain

$$P(1)=-(1-\alpha)\left[1-\frac{(1+p\beta+\gamma)\rho}{\gamma}\right],$$

$$P(-1)=1+2\phi+\alpha+(1-\alpha)\frac{(1+p\beta+\gamma)\rho}{\gamma}>0.$$

From Assumption 1, $P(1)<0$ holds. These results show that $-1<\nu_1<1<\nu_2$ and that the point $(\bar{A}, \bar{N})$ is a saddle point.

B. Existence condition of $\hat{A}$

From (32), the existence condition of $\hat{A}$ is $\Lambda(b)>0$, where $\Lambda(b)\equiv(1-b)\gamma-(1+p\beta+\gamma)\rho(b)$. Here, $\Lambda(b)$ implies

$$\Lambda'(b)=-\gamma-(1+p\beta+\gamma)\rho'(b), \tag{B.1}$$

$$\Lambda''(b)=-(1+p\beta+\gamma)\rho''(b)<0, \tag{B.2}$$

$$\Lambda(1)=-(1+p\beta+\gamma)\rho(1)<0. \tag{B.3}$$

Assuming $\Lambda(0)>0$, we have $\Lambda(b)>0$ when $b<\hat{b}$ and $\Lambda(b)\leq 0$ when $b\geq\hat{b}$, where $\hat{b}$ is defined as $\Lambda(\hat{b})=0$. To ensure the existence of $\hat{A}$, we assume $0\leq b<\hat{b}$.

C. $N_t=\Gamma(A_t)$ locus

We consider the $N_t=\Gamma(A_t)$ locus on the (A_t, N_t) plane. Differentiating $\Gamma(A_t)$ with respect to A_t, we obtain

$$\begin{aligned}\Gamma'(A_t)=&\frac{(1-\phi)(1-z)[(1-z)(1+p\beta)+\gamma]\delta^2 A_t^{-\phi}}{p\beta\bar{\theta}(1-b)[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]^2}\\ &+\frac{(1-\phi)(1+p\beta+\gamma)[\rho(b)\tilde{\alpha}]^2 A_t^{2-2\alpha-\phi}}{p\beta\bar{\theta}(1-b)[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]^2}\\ &+\frac{\Theta\gamma\rho(b)\tilde{\alpha}A_t^{1-\alpha-\phi}}{p\beta\bar{\theta}(1-b)[(1-z)\delta+\rho(b)\tilde{\alpha}A_t^{1-\alpha}]^2},\end{aligned} \tag{C.1}$$

where $\Theta\equiv 2(1-\phi)(1+p\beta+\gamma)-z[2(1-\phi)(1+p\beta)+(2-\alpha-\phi)\gamma]$. The signs of the first and second lines of (C.1) are positive and that of the third line of (C.1) is ambiguous. The condition that the third line of (C.1) becomes positive is given by

$$z\leq\frac{2(1-\phi)(1+p\beta+\gamma)}{2(1-\phi)(1+p\beta)+(2-\alpha-\phi)\gamma}\equiv\tilde{z}.$$

In addition, we have $\tilde{z}\gtreqless 1$ when $\alpha\gtreqless\phi$. Hence, $\Gamma'(A_t)>0$ holds if we assume $z\leq\min\{\tilde{z}, 1\}$.

D. Sign of $\partial\widehat{A}/\partial b$

From (33) and the definition of $\Lambda(b)$, we obtain $\partial\widehat{A}/\partial b\lesseqgtr 0$ when $\Lambda'(b)\gtreqless 0$. Note that Assumption 2 is equal to $\Lambda(0)>0$ and that Assumption 4 is equal to $\Lambda'(0)>0$. Using these assumptions, (B.1), (B.2), and (B.3), we summarize the characteristics of $\Lambda(b)$ as follows:

$$\Lambda(0)>0, \quad \Lambda(1)<0, \quad \Lambda'(0)>0, \quad and \quad \Lambda''(b)<0.$$

Therefore, we obtain

$$\begin{aligned}
&\Lambda(b)>0 \ and \ \Lambda'(b)>0 \ if \ 0\leq b<\tilde{b} \\
&\Lambda(b)>0 \ and \ \Lambda'(b)<0 \ if \ \tilde{b}<b<\hat{b} \\
&\Lambda(b)<0 \ and \ \Lambda'(b)<0 \ if \ \hat{b}<b<1,
\end{aligned}$$

where $\tilde{b}$ is defined as $\Lambda'(\tilde{b})=0$, that is, $\gamma+(1+p\beta+\gamma)\rho'(\tilde{b})=0$. As a result, we can show that $\tilde{b}<\hat{b}$ and $\partial\widehat{A}/\partial b\lesseqgtr 0$ if $b\lesseqgtr\tilde{b}$.

E. Derivation of the growth rates along the BGP

Using (30), we obtain the employment share of R&D along the BGP as follows:

$$l_A^*=\frac{p\beta(1-b)}{1+p\beta+\gamma}-\frac{1}{\theta}\frac{A_t^{1-\phi}}{N_t}.$$

Because the employment share of R&D is constant, the term $A_t^{1-\phi}/N_t$ also becomes constant along the BGP. Thus, the growth rate of A_t is as follows:

$$(1+g_A^*)^{1-\phi}=\left(\frac{A_{t+1}}{A_t}\right)^{1-\phi}=\frac{N_{t+1}}{N_t}=n^*.$$

This shows that g_A^* is determined by n^*. Moreover, from (35), we obtain

$$l_Y^*=\frac{1-\alpha}{1-\alpha+\alpha^2}[1-b-\rho(b)n^*-l_A^*].$$

Because the employment share of the final goods production becomes constant, the growth rate of $L_{Y,t}$ along the BGP is equal to n^*. As a result, g_y^* is given by

$$1+g_y^*=(1+g_A^*)^{1-\alpha}=(n^*)^{\frac{1-\alpha}{1-\phi}}.$$

F. Effect of b on n^*

From (36), we differentiate n^* with respect to b as follows:

$$\frac{\partial n^*}{\partial b}=-\frac{\gamma\chi(b)}{(1+p\beta+\gamma)[\rho(b)]^2},$$

where $\chi(b)\equiv\rho(b)+(1-b)\rho'(b)$. Here, $\chi(b)$ implies

$$\chi'(b)=(1-b)\rho''(b)>0,$$
$$\chi(0)=\rho(0)+\rho'(0)<0,$$
$$\chi(1)=\rho(1)>0.$$

We can show $\chi(0)<0$ from Assumptions 2 and 4. Thus, $\chi(b)\lesseqgtr 0$ holds when $b\lesseqgtr \bar{b}$, where $\bar{b}$ is defined as $\chi(\bar{b})=0$. As a result, we obtain $\partial n^*/\partial b\gtreqless 0$ when $b\lesseqgtr \bar{b}$.

References

Blanchard, O.J. (1985), "Debt, deficits and finite horizons," *The Journal of Political Economy*, 93(2), pp.223–247.

Diamond, P. (1965), "National debt in a neoclassical growth model," *The American Economic Review*, 55(5), pp.1126–1150.

Fukao, K. (2012), *Usinawareta 20 nen to nihonkeizai* (*Lost two decades and the Japanese economy*), in Japanese, Nihon Keizai Shinbunnsha.

Futagami, K. and T. Hori (2010), "Technological progress and population growth : Do we have too few children?" *The Japanese Economic Review*, 61(1), pp.64–84.

Grossman, G. M. and E. Helpman (1991), *Innovation and growth in the global economy*. Cambridge, MA : MIT Press.

Hashimoto, K. and K. Tabata (2016), "Demographic change, human capital accumulation and R&D-based growth," *Canadian Journal of Economics*, 49(2), pp.707–737.

Hirazawa, M. and A. Yakita (2009), "Fertility, child care outside the home, and pay-as-you-go social security," *Journal of Population Economics*, 22(3), pp.565–583.

Jones, C. I. (1995), "R&D-based models of economic growth," *The Journal of Political Economy*, 103(4), pp.759–784.

Nakagawa, M., A. Oura, and Y. Sugimoto (2015), "Notes on economic growth with scale effects : Is depopulation compatible with growth?" *Theoretical Economics Letters*, 5, pp.169–178.

Romer, P. M. (1990), "Endogenous technological change," *The Journal of Political Economy*, 98(5), pp.71–102.

Strulik, H., K. Prettner, and A. Prskawetz (2013), "The past and future of knowledge-based growth," *Journal of Economic Growth*, 18(4), pp.411–437.

Yaari, M. (1965), "Uncertain lifetime, life insurance, and the theory of the consumer," *The Review of Economic Studies*, 32(2), pp.137–150.